Normalization

Recall: An integral domain is <u>normal</u> if it's integrally closed in its field of fractions. Normality is closely related to factoriality:

Prop: If R is a UFD, it's normal.

Pf: (Identical to $R=k[x]$ case) Take $r,s \in R$ relatively prime.

Suppose $\left(\frac{r}{s}\right)^n + a_1\left(\frac{r}{s}\right)^{n-1} + \cdots + a_n = 0.$

$$\Rightarrow r^n = s\left(-a_1 r^{n-1} - \cdots - a_n s^{n-1}\right)$$

So $s \mid r^n$, which contradicts rel. primeness.

$\Rightarrow s$ is a unit in $R \Rightarrow \frac{r}{s} \in R$. $\square$

Cor: The only rational zeros of polynomials over $\mathbb{Z}$ are in $\mathbb{Z}$.

Cor: R a UFD $\Rightarrow R[x_1,\ldots,x_n]$ is normal.

If $R \subseteq S$ are rings and $f \in R[x]$ monic, then f having a root α is the same as $x-\alpha \mid f$ in S. i.e. f has a divisor whose coefficients are integral $/R$. A more general statement holds:

Prop: If f factors in $S[x]$ as $f = gh$, g and h monic, then the coefficients of g and h are integral over R.

Corollary: If $f \in \mathbb{Z}[x]$ is irreducible, then it is irreducible in $\mathbb{Q}[x]$.

Pf of prop: Let $S[x]/(g) = S[\alpha_1]$, α_1 a root of g.
Then by long division, we get $g = (x - \alpha_1) g_1$

Repeating this w/ g_1, and then w/ h, we get an extension ring T of S and elements α_i, β_j of T s.t.

$$g = \prod (x - \alpha_i) \text{ and } h = \prod (x - \beta_j) \text{ in } T[x].$$

So the α_i and β_j are integral over R so the coeffients (in S) are too. $\square$

If R is an integral domain, then if f is monic and $f = gh$, then the leading coefficients must be units, and we get the following:

Cor: If R is normal, then any monic irreducible polynomial $f \in R[x]$ is prime.

Pf: let Q be the field of fractions of R. If $f = gh$ in $Q[x]$, Then $g, h \in R[x]$ (by Prop), so f is also irreducible in $Q[x]$.

Q is a field $\Rightarrow$ $Q[x]$ is a UFD $\Rightarrow$ $(f) \subseteq Q[x]$ is prime.

We also know from a previous Thm that $R[x]/(f)$ is a f.g. free R-module. Thus

$$R[x]/(f) \longrightarrow Q \otimes_R R[x]/(f) \cong Q[x]/(f)$$

is the direct sum of maps $R \longrightarrow Q \otimes_R R = Q$, so it's injective, so $R[x]/(f)$ is an integral domain, so f is prime. □

An important property (geometrically, especially) of normalization is that it commutes w/ localization.

[In particular, if we have a scheme w/ an open cover by affines (of the form Spec A), we can "normalize" the scheme by normalizing each A, and the "normalized" affine schemes will still glue together to form a scheme.]

Prop: $R \subseteq S$ rings, $U \subseteq R$ mult. closed subset. Let S' be the integral closure of R in S. Then $U^{-1}S'$ is the integral closure of $U^{-1}R$ in $U^{-1}S$.

Pf: An element of S integral over R is also integral over

$U^{-1}R$, so $U^{-1}S'$ is integral over $U^{-1}R$.

So we just need to show that if $s/u \in U^{-1}S$ is integral over $U^{-1}R$, then $s/u \in U^{-1}S'$.

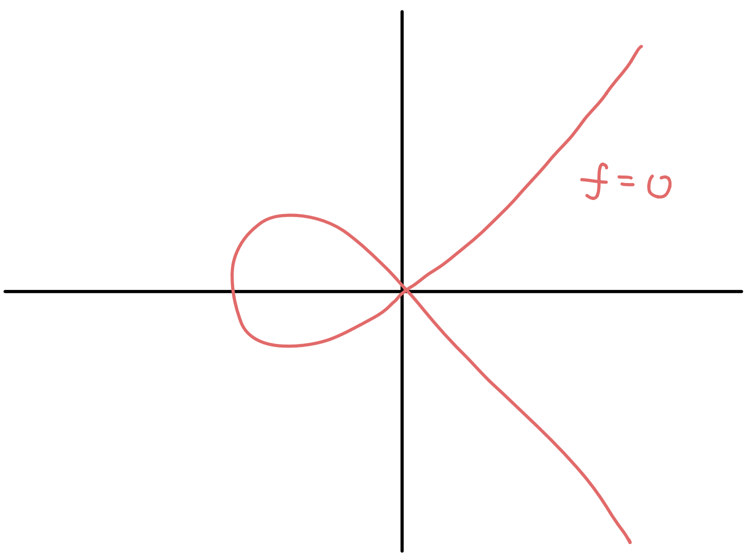
If $\left(\frac{s}{u}\right)^n + \left(\frac{r_1}{u_1}\right)\left(\frac{s}{u}\right)^{n-1} + \cdots + \left(\frac{r_n}{u_n}\right) = 0$, then multiplying by $(u u_1 u_2 \cdots u_n)^n$ yields

$$(u_1 u_2 \cdots u_n s)^n + (r_1 u u_2 u_3 \cdots u_n)(u_1 \cdots u_n s)^{n-1} + \cdots + (r_n u_1^n u_2^n \cdots u_{n-1}^n u_n^{n-1}) = 0.$$

So $u_1 u_2 \cdots u_n s$ is integral over r and is thus in S', so

$$\frac{s}{u} = \frac{u_1 u_2 \cdots u_n s}{u_1 u_2 \cdots u_n u} \in U^{-1}S'. \quad \square$$

Ex: Consider $f = y^2 - x^2(x+1) \in \mathbb{C}[x,y]$.



Let $R = \mathbb{C}[x,y]/(f)$.

The closed (real) points of Spec R

R is not normal:

Define $p(t) = t^2 - (x+1) \in R[t]$.

y/x is in the field of fractions of R but not in R, but

$$P(y/x) = (y/x)^2 - (x+1) = \frac{y^2 - x^2(x+1)}{x^2} = 0.$$

Notice that if we try to evaluate y/x on the zero locus of f, it is defined away from the origin. However, if we take the limit along one branch it is 1 and along the other it's -1. That is, the normalization "separates" the two branches. (See HW #4 for details.)